

VEECH HOLOMORPHIC FAMILIES OF RIEMANN SURFACES, HOLOMORPHIC SECTIONS, AND DIOPHANTINE PROBLEMS

YOSHIHIKO SHINOMIYA

ABSTRACT. In this paper, we construct holomorphic families of Riemann surfaces from Veech groups and characterize their holomorphic sections by some points of corresponding flat surfaces. The construction gives us concrete solutions for some Diophantine equations over function fields. Moreover, we give upper bounds of the numbers of holomorphic sections of certain holomorphic families of Riemann surfaces.

1. INTRODUCTION

A holomorphic family (M, π, B) of Riemann surfaces of type (g, n) is a triple of a two-dimensional complex manifold M , a Riemann surface B and a holomorphic map $\pi : M \rightarrow B$ such that the fiber $X_t = \pi^{-1}(t)$ is a Riemann surface of type (g, n) for each $t \in B$ and the complex structure of X_t depends holomorphically on the parameter t . Holomorphic families of Riemann surfaces are first studied in algebraic geometry as the Diophantine problems for function fields on compact Riemann surfaces.

Let $M(B_0)$ be the field of all meromorphic functions on a compact Riemann surface B_0 . We take a three-variable irreducible homogeneous polynomial f whose coefficients are in $M(B_0)$. The Diophantine problem is to find all solutions $[X : Y : Z] \in \mathbb{P}^2(M(B_0))$ of the Diophantine equation $f(X, Y, Z) = 0$. It is known that a Diophantine equation defines holomorphic family of Riemann surfaces and all the solutions of a Diophantine equation correspond to all the holomorphic sections of the corresponding holomorphic family of Riemann surfaces. By using Teichmüller theory, Iwayoshi and Shiga [IS88] proved that the number of holomorphic sections of a locally non-trivial holomorphic family of Riemann surfaces is finite. And Shiga [Shi97] estimated the number of holomorphic families of Riemann surfaces of certain types. In this paper, we consider holomorphic families of Riemann surfaces obtained by Veech groups.

Let X be a Riemann surface of type (g, n) . The Veech group $\Gamma(X, u)$ is the group of all elements in $\mathrm{PSL}(2, \mathbb{R})$ which induce affine homeomorphisms on a flat surface (X, u) . A Veech group induces a Teichmüller disk which is a holomorphic isometric embedding of a hyperbolic plane \mathbb{H} into the Teichmüller space of the Riemann surface (X, u) . Veech [Vee89] proved that the mirror Veech group $\bar{\Gamma}(X, u)$ is a representation of the action of mapping class group into the Teichmüller disk defined by (X, u) . And the action is regarded as the action of Fuchsian group $\bar{\Gamma}(X, u)$ into \mathbb{H} . Hence, $\mathbb{H}/\bar{\Gamma}(X, u)$ is holomorphically embedded into the moduli space $M(g, n)$. In this paper, we construct holomorphic families of Riemann surfaces from such embeddings and observe their properties. We call such holomorphic families of

Riemann surfaces *Veech holomorphic families of Riemann surfaces*. Especially, we study their holomorphic sections.

The paper is organized as follows. In section 2, we define fundamental objects that we use in this paper, and observe their properties. In section 3, we define Veech groups and show that orbifolds induced by Veech groups are embedded holomorphically and locally isometrically into moduli spaces. Moreover, a relation between geodesic flows on a flat surface and the Veech group is explained in this section. A construction of Veech holomorphic families of Riemann surfaces is discussed in section 4. Also, representations and monodromies of Veech holomorphic families of Riemann surfaces are observed in this section. Section 5 characterizes holomorphic sections of Veech holomorphic families of Riemann surfaces by some points of corresponding flat surfaces. Also, by applying this characterization, we find all the holomorphic sections of certain Veech holomorphic families of Riemann surfaces. In section 6, we construct examples of Diophantine equations from some Veech holomorphic families of Riemann surfaces. All solutions of the Diophantine equations are given by using the characterization of holomorphic sections. Section 7 gives an upper bound of the number of holomorphic sections for certain Veech holomorphic families of Riemann surfaces. The upper bound depends only on topological quantities of the base Riemann surface B and fiber Riemann surfaces $X_t = \pi^{-1}(t)$ of holomorphic families of Riemann surfaces (M, π, B) . To obtain the upper bound, we show a certain property of Fuchsian groups (Theorem 7.13), which may have its own interest.

2. PRELIMINARIES

In this section, we see some properties of Teichmüller spaces, Kobayashi pseudometrics, Teichmüller modular groups, Bers fiber spaces and Fuchsian groups.

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$.

2.1. Teichmüller spaces. Let us consider all pairs (Y, f) of Riemann surfaces Y of type (g, n) and quasiconformal maps f from X onto Y . Two such pairs (Y_1, f_1) and (Y_2, f_2) are said to be equivalent if there exists a conformal map $h : Y_1 \rightarrow Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$. This equivalence relation is called the Teichmüller equivalence.

Definition 2.1 (Teichmüller space). The Teichmüller space $T(X)$ of X is the set of all equivalence classes of such pairs. We denote by $[Y, f]$ the equivalence class of a pair (Y, f) . The point $[X, id : X \rightarrow X]$ is called the base point of $T(X)$.

For any two points $[Y_1, f_1], [Y_2, f_2] \in T(X)$, we set

$$d_T([Y_1, f_1], [Y_2, f_2]) = \inf_h \log K(h).$$

Here, the infimum is taken over all quasiconformal maps $h : Y_1 \rightarrow Y_2$ which are homotopic to $f_2 \circ f_1^{-1}$ and $K(h)$ is the dilatation of h . That is, let $\mu_h = \bar{\partial}h/\partial h$ be the Beltrami coefficient of h . By the definition of quasiconformal maps, the essential supremum norm $\|\mu_h\|_\infty$ of h is less than 1. The dilatation $K(h)$ of h is defined by $K(h) = \frac{1+\|\mu_h\|_\infty}{1-\|\mu_h\|_\infty}$. Then d_T becomes a metric on $T(X)$.

Definition 2.2 (Teichmüller metric). The metric d_T is called the Teichmüller metric.

We consider geodesics of $T(X)$ with respect to the Teichmüller metric d_T .

Definition 2.3 (Holomorphic quadratic differential). A holomorphic quadratic differential q on X is a tensor whose restriction to every coordinate neighborhood (U, z) is the form $f dz^2$, here f is a holomorphic function on U .

We define $|q|$ to be the differential 2-form on X whose restriction to every coordinate neighborhood (U, z) has the form $|f| dx dy$ if q equals $f dz^2$ in U . We say q is integrable if its norm

$$\|q\| = \iint_X |q|$$

is finite. Let \bar{X} be the compact Riemann surface of genus g obtained by filling all punctures of X . We construct a function $\text{ord} : \bar{X} \rightarrow \mathbb{Z}$. If $z \in \bar{X}$ is a zero of q of order n , we set $\text{ord}(z) = n$. If $z \in \bar{X}$ is a pole of q of order n , we set $\text{ord}(z) = -n$. And, we set $\text{ord}(z) = 0$ if $z \in \bar{X}$ is not a zero or pole of q .

Remark. Note that q is integrable if and only if $\text{ord}(z) \geq -1$ for all $z \in \bar{X}$. And, if $q \neq 0$, then by the Riemann-Roch theorem, we have

$$\sum_{z \in \bar{X}} \text{ord}(z) = 4g - 4.$$

Definition 2.4 (Teichmüller map). A quasiconformal map $f : X \rightarrow Y$ is said to be a Teichmüller map if its Beltrami coefficient μ_f is the form

$$\mu_f = k \frac{|q|}{q}$$

for some integrable holomorphic quadratic differential q on X and k ($0 \leq k < 1$).

The next two theorems claim that there is an unique Teichmüller map from X to Y realizing the distance between the base point $[X, id]$ and each point $[Y, f]$.

Theorem 2.5 (Teichmüller's uniqueness theorem). *Let $f_0 : X \rightarrow Y$ be a Teichmüller map and $\mu_{f_0} = k \frac{|q|}{q}$ for some integrable holomorphic quadratic differential q on X and k ($0 \leq k < 1$). Then, for every quasiconformal map $f : X \rightarrow Y$ which is homotopic to f_0 , the inequality*

$$\|\mu_f\|_\infty \geq \|\mu_{f_0}\|_\infty = k$$

holds. Where, μ_f is the Beltrami coefficient of f . The equality holds if and only if $f = f_0$.

Theorem 2.6 (Teichmüller's existence theorem). *Let $f : X \rightarrow Y$ be a quasiconformal map. There exists a Teichmüller map $f_0 : X \rightarrow Y$ which is homotopic to f .*

Therefore, we conclude that the Teichmüller distance between the base point $[X, id]$ and each point $[Y, f]$ is

$$d_T([X, id], [Y, f]) = \log K(f_0) = \log \frac{1+k}{1-k}$$

if the Beltrami coefficient μ_{f_0} of the unique Teichmüller map $f_0 : X \rightarrow Y$ which is homotopic to f is the form $k \frac{|q|}{q}$. For each geodesic ray $r : [0, \infty) \rightarrow T(X)$ with $r(0) = [X, id]$, there exists an unique holomorphic quadratic differential q on X such that $r(t) = [Y_t, f_t]$ if we set $f_t : X \rightarrow Y_t$ to be the Teichmüller map with the Beltrami coefficient $\mu_{f_t} = \frac{t-1}{t+1} \frac{|q|}{q}$. Therefore, d_T is a complete metric. Since the complex dimension of the Banach space of all holomorphic quadratic differentials on X is $3g - 3 + n$, $T(X)$ is homeomorphic to \mathbb{C}^{3g-3+n} and hence, $T(X)$ is proper. For the proof of them, see [IT92].

The Teichmüller space $T(X)$ of X has a complex structure induced by the Bers embedding which is an embedding of $T(X)$ into a certain Banach space of holomorphic quadratic differentials on X . The Teichmüller space is embedded into the Banach space as a bounded domain. See [IT92].

2.2. Kobayashi pseudo-metrics. On a complex manifold, we can define a pseudo-metric which is called the Kobayashi pseudo-metric ([Kob05]). Let M be a connected complex manifold. Given two points $z, w \in M$, we choose points $z = z_0, z_1, \dots, z_{k-1}, z_k = w$ of M , points $a_1, b_1, \dots, a_k, b_k$ of a unit disk \mathbb{D} , and holomorphic maps f_1, \dots, f_k from \mathbb{D} to M with $f_i(a_i) = z_{i-1}$ and $f_i(b_i) = z_i$ for $i = 1, \dots, k$. For each choice of points and maps as above, we consider the number

$$d_{\mathbb{D}}(a_1, b_1) + \dots + d_{\mathbb{D}}(a_k, b_k),$$

where $d_{\mathbb{D}}$ is the hyperbolic metric on \mathbb{D} . Let $d_M^K(z, w)$ be the infimum of the numbers obtained as above. It is easy to see that $d_M^K : M \times M \rightarrow \mathbb{R}$ is continuous and satisfies the axioms of pseudo-metric:

$$d_M^K(z, w) \geq 0, \quad d_M^K(z, w) = d_M^K(w, z) \quad \text{and} \quad d_M^K(z, r) + d_M^K(r, w) \geq d_M^K(z, w).$$

Definition 2.7 (Kobayashi metric, Kobayashi hyperbolic manifold). The pseudo-metric d_M^K on M is called the Kobayashi pseudo-metric of M . If d_M^K satisfies the axioms of metric, we call d_M^K the Kobayashi metric of M , and M a Kobayashi hyperbolic manifold. If a complex manifold M is a Kobayashi hyperbolic manifold and the Kobayashi metric d_M^K is proper, we call M a proper Kobayashi hyperbolic manifold.

By definition, Kobayashi pseudo-metric has a distance decreasing property. That is, for complex manifolds M, N and a holomorphic map $f : M \rightarrow N$, we have

$$d_M^K(z, w) \geq d_N^K(f(z), f(w))$$

for all $z, w \in M$.

Royden [Roy71] prove the following theorem.

Theorem 2.8. *On the Teichmüller space $T(X)$ of X , the Kobayashi pseudo-metric $d_{T(X)}^K$ coincides with the Teichmüller metric d_T .*

This implies that $T(X)$ is a proper Kobayashi hyperbolic manifold.

Remark. If M is a hyperbolic Riemann surface, d_M^K coincides with the complete hyperbolic metric which is compatible with its complex structure.

Proposition 2.9 ([Kob05]). *Let M be a complex manifold and $\pi : \widetilde{M} \rightarrow M$ be a covering map. Choose any two points $z, w \in M$ and fix $\tilde{z} \in \pi^{-1}(z)$. Then*

$$d_M^K(z, w) = \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

Proof. Since π is holomorphic, we have

$$d_M^K(z, w) \leq \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

Assume that there exists $\epsilon > 0$ such that

$$d_M^K(z, w) + \epsilon < \inf_{\tilde{w} \in \pi^{-1}(w)} d_{\widetilde{M}}^K(\tilde{z}, \tilde{w}).$$

By the definition of Kobayashi pseudo-metric, there exist $a_1, b_1, \dots, a_k, b_k \in \mathbb{D}$ and holomorphic maps f_1, \dots, f_k such that

$$z = f(a_1), f(b_i) = f(a_{i+1}) \ (i = 1, \dots, k-1), f_k(a_k) = w$$

and

$$d_M^K(z, w) + \epsilon > \sum_{i=1}^k d_{\mathbb{D}}(a_i, b_i).$$

We lift f_1, \dots, f_k to holomorphic maps $\tilde{f}_1, \dots, \tilde{f}_k$ from \mathbb{D} to \tilde{M} such that $\tilde{z} = \tilde{f}_1(a_1)$ and $\tilde{f}(b_i) = \tilde{f}(a_{i+1}) \ (i = 1, \dots, k-1)$. If we set $\tilde{w} = \tilde{f}_k(b_k)$, then $\pi(\tilde{w}) = w$ and

$$d_{\tilde{M}}^K(\tilde{z}, \tilde{w}) \leq \sum_{i=1}^k d_{\mathbb{D}}(a_i, b_i).$$

Hence, $d_{\tilde{M}}^K(\tilde{z}, \tilde{w}) < d_M^K(z, w) + \epsilon$. This contradicts the assumption. \square

2.3. Teichmüller modular groups. Let $\text{Mod}(X)$ be the group of all homotopy classes of quasiconformal self-maps of X . We denote by $[h]$ the equivalence class of a quasiconformal self-map h of X .

Definition 2.10. The group $\text{Mod}(X)$ is called the the Teichmüller modular group of X .

The Teichmüller modular group $\text{Mod}(X)$ acts on $T(X)$ as follows. For each $[h] \in \text{Mod}(X)$, a map $[h]_* : T(X) \rightarrow T(X)$ is defined by

$$[h]_*[Y, f] = [Y, f \circ h^{-1}]$$

for all $[Y, f] \in T(X)$. Clearly, this action is isometric with respect to the Teichmüller metric d_T . Moreover, it is known that $[h]_*$ is a holomorphic self-map of $T(X)$ and the action of $\text{Mod}(X)$ is properly discontinuous. See [IT92]. Also, the quotient space $T(X)/\text{Mod}(X)$ coincides with the moduli space $M(g, n)$ of Riemann surfaces of type (g, n) .

2.4. Bers fiber spaces. Let $p : \mathbb{H} \rightarrow X$ be the universal covering map with the covering transformation group G . Let $f : X \rightarrow Y$ be a quasiconformal map with Beltrami coefficient μ_f . We define the Beltrami coefficient $\tilde{\mu}_f$ on $\hat{\mathbb{C}}$ which is the lift of μ_f on \mathbb{H} and equals 0 on $\hat{\mathbb{C}} - \mathbb{H}$. Let $w^{\mu_f} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the unique $\tilde{\mu}_f$ -quasiconformal map which fixes $0, 1, \infty$. Then two domains $w^{\mu_{f_1}}(\mathbb{H})$ and $w^{\mu_{f_2}}(\mathbb{H})$ are equal if two quasiconformal maps $f_1 : X \rightarrow Y_1$ and $f_2 : X \rightarrow Y_2$ are Teichmüller equivalent. We denote by \mathbb{H}^t the domain $w^{\mu_f}(\mathbb{H})$ for $t = [Y, f] \in T(X)$. Then

$$F(G) = \{(t, z) : t \in T(X), z \in \mathbb{H}^t\}$$

is called the Bers fiber space over $T(X)$. We induce the complex structure on $F(G)$ as a submanifold of $T(X) \times (\mathbb{C} - \{0, 1\})$. Bers prove the following.

Theorem 2.11 ([Ber73]). *Let us fix a point $a \in X$ and set $\dot{X} = X - \{a\}$. The Bers fiber space $F(G)$ is biholomorphic to $T(\dot{X})$.*

2.5. Holomorphic families of Riemann surfaces.

Definition 2.12 (Holomorphic family of Riemann surfaces). Let \bar{M} be a two-dimensional complex manifold and A be a one-dimensional analytic subset of \bar{M} or empty. Let B be a Riemann surface. Assume that there exists a proper holomorphic map $\bar{\pi} : \bar{M} \rightarrow B$ satisfying the following two conditions.

- (1) By setting $M = \bar{M} - A$ and $\pi = \bar{\pi}|_M$, the holomorphic map π is of maximal rank at every point of M .
- (2) The fiber $X_t = \pi^{-1}(t)$ over each $t \in B$ is a Riemann surface of fixed finite type (g, n) .

We call such triple (M, π, B) a holomorphic family of Riemann surfaces of type (g, n) over B . We assume that $3g - 3 + n > 0$.

Let (M, π, B) be a holomorphic family of Riemann surfaces of type (g, n) . Let $M(g, n)$ be the moduli space of Riemann surfaces of type (g, n) . Then we have a holomorphic map $\phi : B \ni t \mapsto X_t = \pi^{-1}(t) \in M(g, n)$. Let $p : \mathbb{H} \rightarrow B$ be the universal covering map with the covering transformation group $\Gamma < \text{PSL}(2, \mathbb{R})$. We fix $t_0 \in B$ and set $X = \pi^{-1}(t_0)$. Since $M(g, n) = T(X)/\text{Mod}(X)$, we have a lift $\Phi : \mathbb{H} \rightarrow T(X)$ of ϕ . The holomorphic map Φ induces a homomorphism $\chi : \Gamma \rightarrow \text{Mod}(X)$ such that $\Phi \circ A = \chi(A) \circ \Phi$ for every $A \in \Gamma$.

Definition 2.13 (Representation and monodromy). We call the holomorphic map Φ a representation of (M, π, B) into $T(X)$ and the homomorphism χ the monodromy with respect to Φ .

Definition 2.14 (Locally triviality and locally non-triviality). A holomorphic family of Riemann surfaces (M, π, B) is called locally trivial if the induced map $\phi : B \rightarrow M(g, n)$ is constant. And, (M, π, B) is called locally non-trivial if the induced map $\phi : B \rightarrow M(g, n)$ is non-constant.

2.6. Ford region and Shimizu's lemma. Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$ ($\nu_i \in \{2, 3, \dots, \infty\}$). That is, \mathbb{H}/Γ has genus p and cone points whose orders are ν_1, \dots, ν_k . Now, a cone point of order ∞ means a puncture of \mathbb{H}/Γ .

Definition 2.15 (Fundamental domains). A domain D in \mathbb{H} is called a fundamental domain for Γ if it satisfies the followings:

- (1) For all $A \in \Gamma$ ($A \neq id$), $A(D) \cap D = \emptyset$, and
- (2) For all $z \in \mathbb{H}$, there exists $A \in \Gamma$ such that $A(z) \in \text{cl}(D)$.

Here, $\text{cl}(D)$ is the closure of D .

Suppose that Γ contains $\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right]$ and it is not a power of other elements of Γ . We set $\Gamma_0 = \left\{ \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \Gamma : c \neq 0 \right\}$, and for all $A \in \Gamma_0$,

$$D_A = \{z \in \mathbb{H} : |z + \frac{d}{c}| > \frac{1}{|c|}\}.$$

Definition 2.16 (Ford regions). The region

$$D = \{z \in \mathbb{H} : |\text{Re}(z)| < \frac{1}{2}\} \cap \bigcap_{A \in \Gamma_0} D_A$$

is called the Ford region of Γ .

Proposition 2.17. *The Ford region D is a fundamental domain for Γ .*

For a proof of proposition 2.17, see [For25].

Remark. It is clear that the hyperbolic area of \mathbb{H}/Γ induced by the universal covering map $\mathbb{H} \rightarrow \mathbb{H}/\Gamma$ equals to that of the Ford region D in \mathbb{H} . And it is known that the Ford region D becomes a finite sided geodesic polygon in \mathbb{H} .

Lemma 2.18 (Shimizu's lemma). *For all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0$, the inequality $|c| \geq 1$ holds.*

For a proof, see [IT92].

3. FLAT SURFACE AND VEECH GROUPS

In this section, we define Veech groups and see some applications of Veech groups. Let X be a Riemann surface of type (g, n) and C be a set of all punctures of X and finitely many points of X .

Definition 3.1 (Flat structure and flat surface). A flat structure u on X with critical points on C is an atlas of $X' = X - C$ which satisfies the following conditions.

- (1) Local coordinates of u are compatible with the orientation on X induced by its complex structure.
- (2) For coordinate neighborhoods (U, z) and (V, w) of u with $U \cap V \neq \emptyset$, the transition function is the form

$$w = \pm z + c$$

in $z(U \cap V)$ for some $c \in \mathbb{C}$.

- (3) The atlas u is maximal with respect to (1) and (2).

A pair (X, u) of a Riemann surface X and a flat structure u with critical points on C is called a flat surface with critical points on C . Each point in C is called a critical point of (X, u) . In this paper, we assume that the area $\text{Area}(X, u)$ of X with respect to u is finite.

We construct a flat structure u_q on X which is compatible with the complex structure of X from a holomorphic quadratic differential q on X . We set C to be the set of all punctures of X and all zeros of q . For each $p_0 \in X' = X - C$, we can choose an open neighborhood U such that

$$z(p) = \int_{p_0}^p \sqrt{q}$$

is well-defined and an injective function in U . This function is holomorphic in U since q is a holomorphic quadratic differential on X . If (U, z) and (V, w) are pairs of such neighborhoods and functions with $U \cap V \neq \emptyset$, then we have $dw^2 = q = dz^2$ in $U \cap V$. Hence, $w = \pm z + c$ in $z(U \cap V)$ for some $c \in \mathbb{C}$. The flat structure u_q is the maximal flat structure which contains such pairs. The set C is the critical point set of u_q . And the area $\text{Area}(X, u_q)$ equals to the norm $\|q\|$ of q . Conversely, if $u = \{(U_\lambda, z_\lambda)\}$ is a flat structure on X which is compatible with the complex structure of X and the critical point set of u is C , then $q = \{dz_\lambda^2\}$ is a meromorphic quadratic differential on X with $\|q\| = \text{Area}(X, u)$. All zeros and poles of q are contained in C . If $p \in C$ is a zero of q with $\text{odr}(p) = n$, then the angle around p with respect to u is $(n+2)\pi$. And the angle around a single pole of q with respect to u is π .

Fix an integrable holomorphic quadratic differential q on X . Let $u = u_q = \{(U_\lambda, z_\lambda)\}$ be the flat structure induced by q and C be the critical point set of u . For each $A \in \text{SL}(2, \mathbb{R})$, we denote by T_A the linear map on $\mathbb{C} = \mathbb{R}^2$ corresponding to A . Then $u_A = \{(U_\lambda, T_A \circ z_\lambda)\}$ is also a flat structure on X with critical points on C . Note that (X, u_A) is not conformal equivalent to the original Riemann surface X in general. We obtain a map $\hat{i} : \text{SL}(2, \mathbb{R}) \rightarrow T(X)$ such that $\hat{i}(A) = [(X, u_A), \text{id} :$

$X \rightarrow (X, u_A)$. Here, we consider u_A as a new complex structure on X . By the following lemma, $\hat{\iota}$ induces a map $\iota : \mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R}) \rightarrow T(X)$.

Lemma 3.2. $\hat{\iota}(A) = \hat{\iota}(B)$ if $BA^{-1} \in \mathrm{SO}(2)$.

Proof. Let $A, B \in \mathrm{SL}(2, \mathbb{R})$ with $BA^{-1} \in \mathrm{SO}(2)$. For each $(U, z) \in u$, $(T_B \circ z) \circ (T_A \circ z)^{-1} = T_B \circ T_{A^{-1}} = T_{BA^{-1}}$ is a conformal map. Hence, $id : (X, u_A) \rightarrow (X, u_B)$ is a conformal map. \square

The identification of $\mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R})$ with \mathbb{H} by the bijective map

$$\mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbb{R}) \ni \mathrm{SO}(2) \cdot A \mapsto \overline{A^{-1}(i)} \in \mathbb{H}$$

induces a map $\iota : \mathbb{H} \rightarrow T(X)$. Here, $A^{-1}(\cdot)$ is a Möbius transformation. It is known that this map ι is a holomorphic isometric embedding from a hyperbolic plane \mathbb{H} into the Teichmüller space $T(X)$ with the Teichmüller distance. (See [HS07].) And ι satisfies $\iota(i) = [X, id]$. Moreover, it is easy to verify that the Beltrami coefficient of the Teichmüller map from $\iota(i)$ to $\iota(t)$ is $\frac{i-t}{i+t} \frac{|q|}{q}$. Conversely, we have the following.

Proposition 3.3. *Every holomorphic isometric embedding $\iota : \mathbb{H} \rightarrow T(X)$ with $\iota(i) = [X, id]$ is constructed from some flat structure as above.*

Proof. Let $\iota : \mathbb{H} \rightarrow T(X)$ be a holomorphic isometric embedding. Then the geodesic ray $r(t) = it$ ($t \geq 1$) in \mathbb{H} maps to a geodesic ray in $T(X)$.

By theorem 2.5 and theorem 2.6, there exists a holomorphic quadratic differential q on X such that the Teichmüller map from $[X, id]$ to $\iota \circ r(t)$ has the Beltrami coefficient $\frac{1-t}{1+t} \frac{|q|}{q}$ for all $t \geq 1$. Note that these Teichmüller maps are deformations by $\begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix}$ of the flat structure u_q defined by q . Let $\iota_q : \mathbb{H} \rightarrow T(X)$ be the holomorphic isometric embedding constructed from u_q . Since $\iota(it) = \iota_q(it)$ for all $t \geq 1$, we conclude that $\iota = \iota_q$. \square

Definition 3.4 (Teichmüller disk). We call the image of a holomorphic isometric embedding of \mathbb{H} into $T(g, n)$ a Teichmüller disk.

Let (X, u) be a flat surface with critical points on C . Let $\iota : \mathbb{H} \rightarrow T(X)$ be a holomorphic isometric embedding constructed from (X, u) . We consider the image of the Teichmüller disk $\Delta = \iota(\mathbb{H})$ into the moduli space $M(g, n)$ and hence, the subgroup $\mathrm{Stab}(\Delta)$ of $\mathrm{Mod}(X)$ which consists of all elements of $\mathrm{Mod}(X)$ which fixing Δ . For this, we define the affine group $\mathrm{Aff}^+(X, u)$ of (X, u) .

Definition 3.5 (Affine group of (X, u)). The affine group $\mathrm{Aff}^+(X, u)$ of the flat surface (X, u) is the group of all quasiconformal maps h of X onto itself which satisfy $h(C) = C$ and are affine with respect to the flat structure u . This means that for $(U, z), (V, w) \in u$ with $h(U) \subseteq V$, the homeomorphism $w \circ h \circ z^{-1}$ is the form $z \mapsto Az + c$ for some $A \in \mathrm{GL}(2, \mathbb{R})$ and $c \in \mathbb{C}$. Each element in $\mathrm{Aff}^+(X, u)$ is called an affine homeomorphism on (X, u) .

For each $h \in \mathrm{Aff}^+(X, u)$, the derivative $A \in \mathrm{GL}(2, \mathbb{R})$ of the affine map $w \circ h \circ z^{-1}$ is uniquely determined up to the sign since u is a flat structure. And A is always in $\mathrm{SL}(2, \mathbb{R})$ since $\mathrm{Area}(X, u) = \int_X |q| = \int_X h^*(|q|) = \det(A) \cdot \mathrm{Area}(X, u)$. Here, q is a holomorphic quadratic differential corresponding to (X, u) . Thus, we have a group homomorphism

$$D : \mathrm{Aff}^+(X, u) \rightarrow \mathrm{PSL}(2, \mathbb{R}).$$

We call this homomorphism the derivative map.

Definition 3.6 (Veech group of (X, u)). We call $\Gamma(X, u) = D(\text{Aff}^+(X, u))$ the Veech group of (X, u) .

Veech[Vee89] prove the following.

Proposition 3.7. *All homomorphisms of $\text{Aff}^+(X, u)$ are not homotopic to each other. Therefore, $\text{Aff}^+(X, u)$ is regarded as a subgroup of $\text{Mod}(X)$.*

By setting $\iota(t) = [X_t, id]$ for each $t \in \mathbb{H}$, we have the following theorem.

Theorem 3.8 ([Vee89], [EG97], [HS07]). *The group $\text{Stab}(\Delta)$ coincides with $\text{Aff}^+(X, u)$. For any $h \in \text{Aff}^+(X, u)$ and $[X_t, id] \in \Delta$,*

$$h_*[X_t, id] = [X_t, h^{-1}] = [X_{\bar{A}(t)}, id].$$

Here, $A = D(h)$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\bar{A} = RAR^{-1}$ acts on \mathbb{H} as a Möbius transformation.

Corollary 3.9 ([Vee89], [EG97], [HS07]). *The Veech groups $\Gamma(X, u)$ is a Fuchsian group and we have the following commutative diagram. Here, $\bar{\Gamma}(X, u) = R\Gamma(X, u)R^{-1}$.*

$$\begin{array}{ccccc} \mathbb{H} & \xrightarrow{\iota} & \Delta & \hookrightarrow & T(X) \\ \downarrow / \bar{\Gamma}(X, u) & & \downarrow / \text{Stab}(\Delta) & & \downarrow / \text{Mod}(X) \\ \mathbb{H} / \bar{\Gamma}(X, u) & \xrightarrow{\sim} & \Delta / \text{Stab}(\Delta) & \hookrightarrow & M(g, n) \end{array}$$

We construct holomorphic families of Riemann surfaces from such holomorphic embeddings $\mathbb{H} / \bar{\Gamma}(X, u) \rightarrow M(g, n)$.

Next, we see an application of Veech groups for a dynamical behavior of the geodesic flows on flat surfaces (X, u) . We assume that the conformal structure defined by u is compatible with that of X . Let C be the critical point set of (X, u) . Since the transition functions of (X, u) are the form $z \mapsto \pm z + c$, geodesics on (X, u) and directions of geodesics of (X, u) are well-defined. That is, a geodesic $l : \mathbb{R} \rightarrow X$ on (X, u) is a curve in $X' = X - C$ such that for all $t \in \mathbb{R}$, sufficiently small $\epsilon > 0$, and $(U, z) \in u$ with $l((t - \epsilon, t + \epsilon)) \subset U$, $z \circ l|_{(t - \epsilon, t + \epsilon)}$ is an Euclidean line segment. The direction $\theta(l) \in [0, \pi)$ of a geodesic l is the direction of $z \circ l|_{(t - \epsilon, t + \epsilon)}$ on \mathbb{C} . A saddle connection is a path on X which connects two points in C and is locally an Euclidean line segment with respect to u .

Definition 3.10 (Jenkins-Strebel direction). A number $\theta \in [0, \pi)$ is called a Jenkins-Strebel direction of (X, u) if all points in X lie on θ -direction closed geodesics or θ -direction saddle connections of (X, u) .

If θ is a Jenkins-Strebel direction of (X, u) , the flat surface (X, u) is decomposed into finitely many Euclidean cylinders by cutting along the θ -direction saddle connections.

Definition 3.11 (Simple Jenkins-Strebel direction). If Jenkins-Strebel direction θ decomposes (X, u) into only one cylinder, we call the direction θ a simple Jenkins-Strebel direction of (X, u) .

Definition 3.12. For an Euclidean cylinder R whose length of circumference is b and height is a , the ratio $\frac{b}{a}$ is called the modulus of R and denote it by $\text{mod}(R)$.

Veech[Vee89] proved the following theorem which is called the Veech dichotomy theorem. For the definition of the ergodicity, see [KH96].

Theorem 3.13. *Let (X, u) be a flat surface. Suppose that the Veech group $\Gamma(X, u)$ is a lattice in $\text{PSL}(2, \mathbb{R})$, that is, $\mathbb{H}/\Gamma(X, u)$ has finite hyperbolic area. Then for each direction θ , one of the following two possibilities occurs:*

- *The direction θ is a Jenkins-Strebel direction. Moreover, if θ decomposes X into k cylinders R_1, \dots, R_k , the ratio $\text{mod}(R_i)/\text{mod}(R_j)$ is a rational number for each $i, j \in \{1, \dots, k\}$.*
- *Every θ -direction geodesic is dense in X . And the θ -direction geodesic flow is uniquely ergodic.*

Finally, we see two examples of Veech groups.

Example 3.14 ([Shi11]). Let X be a surface constructed as figure 1. We induce an unique conformal structure on X such that the quadratic differential dz^2 on the interior of the rectangle of figure 1 extends to a holomorphic quadratic differential q on X . Then X is a Riemann surface of type $(2, 0)$ and vertices of four squares become two points in X . These points are zeros of q of order 2. Let $u = u_q$ be the flat structure defined by q . Then $\theta = 0$ and $\frac{\pi}{2}$ are Jenkins-Strebel directions of (X, u) . The direction $\theta = \frac{\pi}{2}$ decomposes (X, u) into two cylinders. And the direction $\theta = 0$ decomposes (X, u) into only one cylinder. Therefore, $\theta = 0$ is a simple Jenkins-Strebel direction. The two Jenkins-Strebel directions $0, \frac{\pi}{2}$ of (X, u) imply that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ define elements in $\text{Aff}^+(X, u)$ as figure 2. Hence, $\Gamma = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle$ is a subgroup of $\Gamma(X, u)$.

Since every element in $\text{Aff}^+(X, u)$ must preserve the set of all critical points, $\Gamma(X, u)$ is a subgroup of $\text{PSL}(2, \mathbb{Z})$. It is known that $\left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle$ is the congruence subgroup of level 2 and has index 6 in $\text{PSL}(2, \mathbb{Z})$. Hence, $\Gamma(X, u)$ is either Γ or $\text{PSL}(2, \mathbb{Z})$. However, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ cannot be an element in $\Gamma(X, u)$. Therefore $\Gamma(X, u)$ must be Γ . It is easy to see that $\mathbb{H}/\Gamma(X, u)$ is an orbifold of genus 0 has 2 punctures and 1 cone point of order 2. Thus, $\Gamma(X, u)$ is a lattice in $\text{PSL}(2, \mathbb{R})$ and the flat surface (X, u) satisfies the Veech dichotomy.

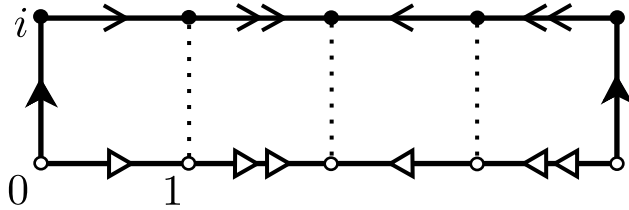
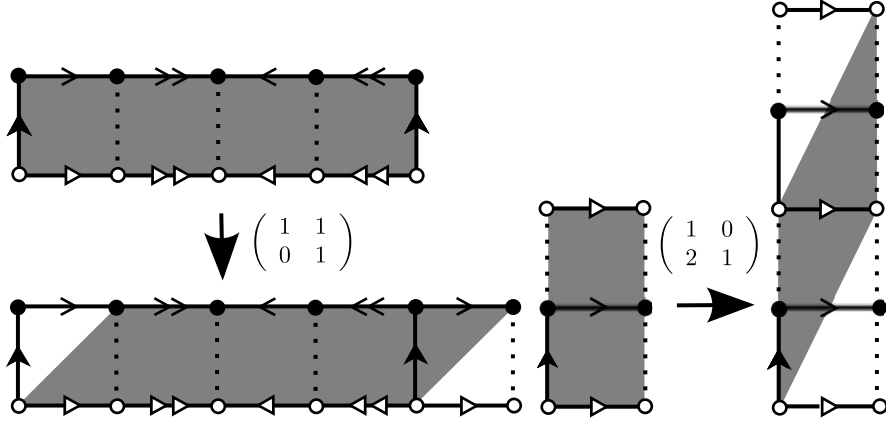
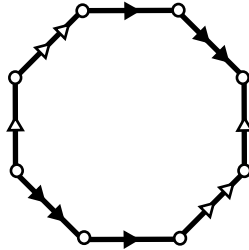


FIGURE 1. The flat surface (X, u)

Example 3.15 ([EG97]). Fix $n \geq 2$ and let Π_n be a regular $4n$ -gon. We assume that Π_n has two horizontal sides, lengths of the sides are 1. We identify each side

FIGURE 2. Actions of two matrices onto X

of Π_n with the opposite parallel side by an Euclidean translation (see figure 3) and denote the resulting surface by P_n . We induce an unique conformal structure on P_n such that the quadratic differential dz^2 on the interior of Π_n extends to a holomorphic quadratic differential q_n on P_n . Then P_n is a Riemann surface of type $(\frac{n}{2}, 0)$. And, the vertices of Π_n become an unique zero of q_n of order $4n - 4$. Let u_{q_n} be the flat structure defined by q_n . Now, $R_n = \begin{pmatrix} \cos \pi/2n & -\sin \pi/2n \\ \sin \pi/2n & \cos \pi/2n \end{pmatrix}$ and $T_n = \begin{pmatrix} 1 & 2 \cot \pi/4n \\ 0 & 1 \end{pmatrix}$ induce elements in $\text{Aff}^+(P_n, u_{q_n})$. The action of R_n on P_n is the rotation about the center of Π_n of angle $\frac{\pi}{2n}$. To see the action of T_n on P_n , we cut P_n along all horizontal saddle connections. Then P_n is decomposed into $\frac{n}{2}$ cylinders and the action of T_n is the composition of the square of the right Dehn twist along a core curve of the cylinder which contains the center of Π_n and the right Dehn twists along core curves of the other cylinders. Thus, $\Gamma = \langle [R_n], [T_n] \rangle$ is a subgroup of the Veech group $\Gamma(P_n, u_{q_n})$. It is easy to see that Γ is a triangle group of type $(2n, \infty, \infty)$. Since only discrete group that contains Γ is a triangle group of type $(2, 4n, \infty)$ (see [EG97] and [Sin72]) and this cannot be $\Gamma(P_n, u_{q_n})$, we have $\Gamma(P_n, u_{q_n}) = \langle [R_n], [T_n] \rangle$. And, $\Gamma(P_n, u_{q_n})$ is a lattice in $\text{PSL}(2, \mathbb{R})$ and the flat surface (P_n, u_{q_n}) satisfies the Veech dichotomy.

FIGURE 3. The flat surface (P_8, u_{q_8})

4. CONSTRUCTION OF VEECH HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

In this section, we construct holomorphic families of Riemann surfaces from flat surfaces and the Veech groups. We call those holomorphic families *Veech holomorphic families of Riemann surfaces*. And we observe representations and monodromies of Veech holomorphic families of Riemann surfaces.

We assume that X is a Riemann surface of type (g, n) with $3g - 3 + n > 0$. Let (X, u) be a flat surface with critical points on C and assume that this flat structure is compatible with the complex structure of X . Let q be the corresponding holomorphic quadratic differential on X . In section 3, we constructed a holomorphic isometric embedding $\iota : \mathbb{H} \ni t \mapsto [X_t, id] \in T(X)$ from (X, u) . The Beltrami coefficient of the Teichmüller map from $\iota(i) = [X, id]$ to $\iota(t)$ is $\mu_t = \frac{i-t}{i+t} \frac{|q|}{q}$. Let $\rho : \mathbb{H} \rightarrow X$ be the universal covering map with the covering transformation group $G < \text{PSL}(2, \mathbb{R})$. For each $t \in \mathbb{H}$, we denote by $\tilde{\mu}_t$ the Beltrami coefficient on $\hat{\mathbb{C}}$ which is the lift of μ_t in \mathbb{H} and equals to 0 in $\hat{\mathbb{C}} - \mathbb{H}$. Let $w^{\tilde{\mu}_t} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a unique $\tilde{\mu}_t$ -quasiconformal map which fixes 0, 1, ∞ . We set

$$\widetilde{M} = \{(t, z) : t \in \mathbb{H}, z \in w^{\tilde{\mu}_t}(\mathbb{H})\}.$$

By identifying \mathbb{H} with $\iota(\mathbb{H})$, \widetilde{M} is a two-dimensional complex manifold as a submanifold of the Bers fiber space $F(G)$. The covering transformation group G acts on \widetilde{M} holomorphically as $\sigma_*(t, z) = (t, w^{\tilde{\mu}_t} \sigma(w^{\tilde{\mu}_t})^{-1}(z))$ for all $\sigma \in G$ and $(t, z) \in \widetilde{M}$. Set $\widehat{M} = \widetilde{M}/G$. Then we can write \widehat{M} as

$$\widehat{M} = \{(t, z) : t \in \mathbb{H}, z \in X_t = f_t(X)\}.$$

Here, $f_t : X \rightarrow X_t = f_t(X)$ is the induced quasiconformal map of $w^{\tilde{\mu}_t}|_{\mathbb{H}}$. Of course, X_t coincides with $\iota(t)$ and f_t is a Teichmüller map with the Beltrami coefficient μ_t .

Recall that $\text{Aff}^+(X, u)$ is the group of all affine homeomorphisms from (X, u) onto itself which fixes C and the homomorphism $D : \text{Aff}^+(X, u) \rightarrow \text{PSL}(2, \mathbb{R})$ is the derivative map. The affine group $\text{Aff}^+(X, u)$ acts on \widehat{M} holomorphically as

$$h_*(t, z) = (\bar{A}(t), f_{\bar{A}(t)} h(f_t)^{-1}(z))$$

for all $h \in \text{Aff}^+(X, u)$ and $(t, z) \in \widehat{M}$. Here, $A = D(h)$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{A} = RAR^{-1}$ acts on \mathbb{H} as a Möbius transformation. We set $N = \widehat{M}/\text{Aff}^+(X, u)$ and $\bar{B} = \mathbb{H}/\bar{\Gamma}(X, u)$. Let $\pi' : N \rightarrow \bar{B}$ be the projection, \mathbb{H}^* be \mathbb{H} with all elliptic fixed points of $\bar{\Gamma}(X, u)$ are removed, $B = \mathbb{H}^*/\bar{\Gamma}(X, u)$, and $N^* = (\pi')^{-1}(B)$. It is easy to see that N is homeomorphic to the product space $B \times (X/\text{Ker}(D))$. There exists a branched covering map $M \rightarrow N$ which corresponds to the branched covering map $B \times X \rightarrow B \times (X/\text{Ker}(D))$. Now, we obtain the holomorphic family (M, π, B) of Riemann surfaces of type (g, n) over B . Here, $\pi : M \rightarrow B$ is the projection.

Definition 4.1 (Veech holomorphic families). We call the holomorphic families of Riemann surfaces (M, π, B) which are constructed as above Veech holomorphic families of Riemann surfaces of type (g, n) over B .

Remark. The triple (N^*, π', B) is also a holomorphic family of Riemann surfaces. And, (M, π, B) and (N^*, π', B) are locally non-trivial holomorphic families of Riemann surfaces.

We observe a representation of a Veech holomorphic family (M, π, B) of Riemann surfaces and the monodromy. First, we have the holomorphic isometric embedding

$\iota : \mathbb{H} \rightarrow T(g, n)$. Let $\bar{\rho} : \mathbb{H} \rightarrow \bar{B}$ and $\rho : \mathbb{H} \rightarrow B$ be the universal covering maps. Since \mathbb{H}^* is a covering space of B , there exists a covering map $\rho_0 : \mathbb{H} \rightarrow \mathbb{H}^*$ which satisfies $\rho = \bar{\rho} \circ \rho_0$. Then, the map $\Phi = \iota \circ \rho_0$ is a representation of (M, π, B) . Since $B = \mathbb{H}^* / \bar{\Gamma}(X, u)$, we can identify $\bar{\Gamma}(X, u)$ with $\pi_1(B) / \pi_1(\mathbb{H}^*)$. And there exists a natural map $\bar{\Gamma}(X, u) \rightarrow \Gamma(X, u)$. The monodromy $\chi : \pi_1(B) \rightarrow \text{Aff}^+(X, u)$ of the representation Φ is a lift of this natural map by the maps $\pi_1(B) \rightarrow \pi_1(B) / \pi_1(\mathbb{H}^*)$ and $D : \text{Aff}^+(X, u) \rightarrow \Gamma(X, u)$.

$$\begin{array}{ccc}
 \mathbb{H} & \xrightarrow{\Phi} & T(X) \\
 \rho_0 \searrow & & \uparrow \iota \\
 \mathbb{H}^* & \xrightarrow{\quad} & \\
 \rho \downarrow & & \downarrow \bar{\rho}|_{\mathbb{H}^*} \\
 B & &
 \end{array}$$

$$\begin{array}{ccc}
 \pi_1(B) & \xrightarrow{\chi} & \text{Aff}^+(X, u) \\
 \downarrow & & \downarrow D \\
 \bar{\Gamma}(X, u) = \pi_1(B) / \pi_1(\mathbb{H}^*) & \xrightarrow{\quad} & \Gamma(X, u)
 \end{array}$$

5. HOLOMORPHIC SECTIONS OF VEECH HOLOMORPHIC FAMILIES OF RIEMANN SURFACES.

In this section, we characterize holomorphic sections of holomorphic families of Riemann surfaces by certain points of a corresponding flat surface. We use the same notations as in section 4.

Definition 5.1 (Holomorphic sections). Let (M, π, B) be a holomorphic family of Riemann surfaces. A holomorphic map $s : B \rightarrow M$ is called a holomorphic section of (M, π, B) or a holomorphic section of π if it satisfies $\pi \circ s = \text{id}$.

Let (M, π, B) be a Veech holomorphic family of Riemann surfaces defined by a holomorphic quadratic differential q or the corresponding flat surface (X, u) . In section 4, we define the action of $\text{Aff}^+(X, u)$ onto \widehat{M} . We set $\widehat{N} = \widehat{M} / \text{Ker}(D)$ and let $\hat{\pi} : \widehat{N} \rightarrow \mathbb{H}$ be the projection. We denote each point of \widehat{N} by (t, w) . Here, $t \in \mathbb{H}$ is the image of the point by the map $\hat{\pi}$ into \mathbb{H} and w is a point of the Riemann surface $\hat{\pi}^{-1}(t) = Y_t$. That is, we denote \widehat{N} by

$$\widehat{N} = \{(t, w) : t \in \mathbb{H}, w \in Y_t\}.$$

Note that if we set q' to be a holomorphic quadratic differential on $Y = X / \text{Ker}(D)$ induced by q , then there exists the Teichmüller map $g_t : Y \rightarrow Y_t$ whose Beltrami coefficient is $\frac{i-t}{i+t} \frac{|q'|}{q'}$ for each $t \in \mathbb{H}$.

Let $s : B \rightarrow M$ be a holomorphic section of (M, π, B) . We project this holomorphic section to a holomorphic section $s' : B \rightarrow N^*$ of (N^*, π', B) . Where, (N^*, π', B) is a holomorphic family of Riemann surfaces constructed in section 4. We lift s' to a holomorphic section $\hat{s} : \mathbb{H}^* \rightarrow \widehat{N}^* = \hat{\pi}^{-1}(\mathbb{H}^*) \subset \widehat{N}$ of $\hat{\pi}|_{\widehat{N}^*}$.

$$\begin{array}{ccccc}
& & & M & \\
& & & \downarrow & \nearrow \\
\widehat{N} \supset \widehat{N}^* & \xrightarrow{\quad} & N^* & & \\
\downarrow \hat{\pi} & \searrow \hat{s} & \downarrow \pi' & & \\
\mathbb{H} \supset \mathbb{H}^* & \xrightarrow{\quad} & B & &
\end{array}$$

Proposition 5.2. *The complex manifold \widehat{N} is a proper Kobayashi hyperbolic manifold.*

Proof. Let \widetilde{N} be the universal covering of \widehat{N} . First, we see that \widetilde{N} is a proper Kobayashi hyperbolic manifold. Recall that q' is the holomorphic quadratic differential on $Y = X/\text{Ker}(D)$ which is the projection of the holomorphic quadratic differential q on X . By setting $\iota' : \mathbb{H} \rightarrow T(Y)$ to be the holomorphic isometric embedding defined by q' . Then we can write \widetilde{N} as

$$\widetilde{N} = \{(\tilde{t}, \tilde{w}) : \tilde{t} \in \mathbb{H}, \tilde{w} \in \mathbb{H}^{\tilde{t}}\}$$

by identifying \mathbb{H} as $\iota'(\mathbb{H})$. And $\mathbb{H}^{\tilde{t}}$ is defined in subsection 2.4. Therefore, \widetilde{N} is a closed submanifold of the Bers fiber space $F(G')$. Here, G' is a Fuchsian group with $Y = \mathbb{H}/G'$. Since $F(G')$ is proper Kobayashi hyperbolic manifold by theorem 2.8 and theorem 2.11, \widetilde{N} is also a proper Kobayashi hyperbolic manifold.

Next, we show that \widehat{N} is a Kobayashi hyperbolic manifold. We suppose that $d_{\widehat{N}}^K((t_1, w_1), (t_2, w_2)) = 0$. Since $\hat{\pi}$ is holomorphic,

$$d_{\mathbb{H}}(t_1, t_2) \leq d_{\widehat{N}}^K((t_1, w_1), (t_2, w_2)) = 0.$$

Hence, we have $t_1 = t_2$. We set $t = t_1 = t_2$. Let $\rho : \widetilde{N} \rightarrow \widehat{N}$ be the universal covering map. Fix $(\tilde{t}, \tilde{w}_1) \in \rho^{-1}((t, w_1))$. By theorem 2.9, we have

$$0 = d_{\widehat{N}}^K((t, w_1), (t, w_2)) = \inf_{(\tilde{t}, \tilde{w}_2)} d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_2)).$$

Here, infimum is taken over all points in $\rho^{-1}((t, w_2))$. For all $n \geq 2$, there exist (\tilde{t}, \tilde{w}_n) such that $d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_n)) < \frac{1}{n}$. Since \widetilde{N} is holomorphically embedded in $\mathbb{H} \times (\mathbb{C} - \{0, 1\})$, we have

$$d_{\mathbb{C}-\{0,1\}}^K(\tilde{w}_1, \tilde{w}_n) \leq d_{\widetilde{N}}^K((\tilde{t}, \tilde{w}_1), (\tilde{t}, \tilde{w}_n)) < \frac{1}{n}.$$

Since the relative topology of \mathbb{H}^t from $d_{\mathbb{C}-\{0,1\}}^K$ is same as the topology defined by $d_{\mathbb{H}^t}^K$ and $w_1 \in \mathbb{H}^t$, we have $d_{\mathbb{H}^t}^K(\tilde{w}_1, \tilde{w}_n) \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\tilde{w}_n = \tilde{w}_{n+1}$ for sufficiently large n and hence, $\tilde{w}_n = \tilde{w}_1$. Therefore, we have $(t_1, w_1) = (t_2, w_2)$.

Finally we prove that \widehat{N} is a proper Kobayashi hyperbolic manifold. Let $U_r(t_0, w_0)$ be the closed ball of center $(t_0, w_0) \in \widehat{N}$ and radius r in \widehat{N} . Take any sequence $\{(t_n, w_n)\}_{n \in \mathbb{N}}$ in $U_r(t_0, w_0)$. We fix $(\tilde{t}_0, \tilde{w}_0) \in \rho^{-1}((t_0, w_0))$. By proposition 2.9, there exists $(\tilde{t}_n, \tilde{w}_n) \in \rho^{-1}((t_n, w_n))$ such that $d_{\widetilde{N}}^K((\tilde{t}_0, \tilde{w}_0), (\tilde{t}_n, \tilde{w}_n)) < r$ for each $n \in \mathbb{N}$. Since \widetilde{N} is proper, there exists a subsequence $\{(\tilde{t}_{n_l}, \tilde{w}_{n_l})\}$ which converges to some point $(\tilde{t}_\infty, \tilde{w}_\infty) \in \widetilde{N}$. Then,

$$d_{\widehat{N}}^K((t_{n_l}, w_{n_l}), \rho(\tilde{t}_\infty, \tilde{w}_\infty)) \leq d_{\widetilde{N}}^K((\tilde{t}_{n_l}, \tilde{w}_{n_l}), (\tilde{t}_\infty, \tilde{w}_\infty)) \rightarrow 0 \quad (l \rightarrow \infty).$$

Hence, \widehat{N} is a proper Kobayashi hyperbolic manifold. \square

By proposition 5.2, we can extend $\hat{s} : \mathbb{H}^* \rightarrow \widehat{N}^*$ to a holomorphic section $\hat{s} : \mathbb{H} \rightarrow \widehat{N}$ of $\hat{\pi}$. For holomorphic sections of $\hat{\pi}$, we have the following. Recall that q' is a holomorphic quadratic differential on $Y = X/\text{Ker}(D)$ induced by q and $g_t : Y \rightarrow Y_t$ is the Teichmüller map whose Beltrami coefficient is $\frac{i-t}{i+t} \frac{|q'|}{q'}$.

Theorem 5.3. *Let $\hat{s} : \mathbb{H} \rightarrow \widehat{N}$ be a holomorphic section of $\hat{\pi}$. Set $b = \hat{s}(i)$. Then*

$$\hat{s}(t) = (t, g_t(b))$$

for all $t \in \mathbb{H}$.

Proof. Set $\dot{Y} = Y - \{b\}$. Let $\iota' : \mathbb{H} \rightarrow T(Y)$ be the holomorphic isometric embedding defined by q' . Let $\tilde{s} : \mathbb{H} \rightarrow \tilde{N}$ be a lift of \hat{s} to \tilde{N} . Since, \tilde{N} is embedded holomorphically into the Bers fiber space $F(G')$, it is also embedded holomorphically into $T(\dot{Y})$. Here, G' is a Fuchsian group with $Y = \mathbb{H}/G'$. The composition of this embedding with \tilde{s} is an injective holomorphic map $\kappa : \mathbb{H} \rightarrow T(\dot{Y})$ with $\kappa(i) = [\dot{Y}, id]$. Let $\tau : T(\dot{Y}) \rightarrow T(Y)$ be the forgetful map, that is, $\tau([f(\dot{Y}), f]) = [f(Y), f]$ for all $[f(\dot{Y}), f] \in T(\dot{Y})$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\quad} & T(\dot{Y}) \\ \tilde{s} \downarrow & \nearrow \kappa & \downarrow \tau \\ \mathbb{H} & \xrightarrow{\iota'} & T(Y) \end{array}$$

Since κ, τ is holomorphic and ι is holomorphic isometric embedding, by theorem 2.8, $\kappa : \mathbb{H} \rightarrow T(\dot{Y})$ is also an isometry. By proposition 3.3, there exists a holomorphic quadratic differential \dot{q} on \dot{Y} which defines κ . Take some $t \in \mathbb{H}$ ($t \neq i$) then the equation $\tau \circ \kappa(t) = \iota'(t)$ implies that $\frac{i-t}{i+t} \frac{|\dot{q}|}{\dot{q}} = \frac{i-t}{i+t} \frac{|q'|}{q'}$ on \dot{Y} . This means that $\hat{s}(t) = (t, g_t(b))$ for all $t \in \mathbb{H}$. \square

Now, we observe that N^* is a quotient manifold of \widehat{N}^* by the action of the Veech group $\Gamma(X, u)$. Let u' be the flat structure on $Y = X/\text{Ker}(D)$ defined by q' . Since $\Gamma(X, u) = \text{Aff}^+(X, u)/\text{Ker}(D)$, the Veech group $\Gamma(X, u)$ is considered as a subgroup of $\text{Aff}^+(Y, u')$ and acts on Y . We denote by h'_A each element A of $\Gamma(X, u)$ if we consider A as an element of $\text{Aff}^+(Y, u')$. We extend the action of $\Gamma(X, u)$ onto Y to an action onto \widehat{N} . For all $(t, w) \in \widehat{N}$ and $h'_A = A \in \Gamma(X, u) < \text{Aff}^+(Y, u')$, we set

$$(h'_A)_*(t, w) = (\bar{A}(t), g_{\bar{A}(t)} h'_A g_t^{-1}(w)).$$

Recall that $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{A} = RAR^{-1}$ acts on \mathbb{H} as a Möbius transformation. This action is induced by the action of $\text{Aff}^+(X, u)$ on \widehat{M} which we define in section 4. Then, it is clear that $N^* = \widehat{N}^*/\text{Ker}(D)$. Now we have the following.

Corollary 5.4. *Let $\varphi : X \rightarrow Y$ be the branched covering map. For $b \in Y$ and any point $a \in \varphi^{-1}(b)$, the following are equivalent.*

- (1) The map $\hat{s}_b : \mathbb{H} \ni t \mapsto (t, g_t(b)) \in \widehat{N}$ induces a section of $\pi' : N^* \rightarrow B$,
- (2) $\Gamma(X, u)(\{b\}) = \{b\}$,
- (3) $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$.

Proof. (1) \Rightarrow (2). If \hat{s}_b induces a section of π' , then $(h'_A)_*(\hat{s}_b(t)) = \hat{s}_b(\bar{A}(t))$ for all $h'_A = A \in \Gamma(X, u)$. This implies $h'_A(b) = b$ for all $A \in \Gamma(X, u)$.

(2) \Rightarrow (1). If $\Gamma(X, u)(\{b\}) = \{b\}$ holds, then $(h'_A)_*(s_b(t)) = s_b(\bar{A}(t))$ for all $h'_A = A \in \Gamma(X, u)$. Hence, we obtain (1).

Clearly, (2) and (3) are equivalent since $\Gamma(X, u) = \text{Aff}^+(X, u)/\text{Ker}(D)$. \square

Recall that $\bar{\rho}|_{\mathbb{H}^*} : \mathbb{H}^* \rightarrow B = \mathbb{H}^*/\bar{\Gamma}(X, u)$ is a covering map and $f_t : X \rightarrow X_t$ is the Teichmüller map with the Beltrami coefficient $\frac{i-t}{i+t} \frac{q}{q}$ for each $t \in \mathbb{H}^*$. Fix any $t \in \mathbb{H}^*$. We take an open neighborhood \tilde{U} of \tilde{t} in \mathbb{H}^* such that $\bar{\rho}|_{\tilde{U}} : \tilde{U} \rightarrow \bar{\rho}(\tilde{U}) = U$ is a conformal map. Then $\pi^{-1}(U) \subset M$ is biholomorphic to $\tilde{M}(\tilde{U}) = \{(t, z) : t \in \tilde{U}, z \in X_t\} \subset \tilde{M}$. By identifying U with \tilde{U} and $\pi^{-1}(U)$ with $\tilde{M}(\tilde{U})$, we have the following.

Corollary 5.5. *Let $s : B \rightarrow M$ be a holomorphic section of π . Then there exists $a \in X$ with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$ such that the induced holomorphic map $s|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{M}(\tilde{U})$ satisfies*

$$s|_{\tilde{U}}(t) = (t, f_t(a))$$

for all $t \in U$.

Fix $\Gamma < \Gamma(X, u)$. Let $\rho' : \mathbb{H}^*/\Gamma \rightarrow B$ be a covering map. Set $M_\Gamma = \{(t', z') : t' \in \mathbb{H}^*/\Gamma, z' \in X_{\rho'(t')} = \pi^{-1}(\rho'(t'))\}$ and let $\pi_\Gamma : M_\Gamma \rightarrow \mathbb{H}^*/\Gamma$ be the projection. Then the triple $(M_\Gamma, \pi_\Gamma, \mathbb{H}^*/\Gamma)$ is a holomorphic family of Riemann surfaces, M_Γ is a covering space of M , and π_Γ is a lift of π . By identifying $V = (\rho')^{-1}(U)$ with \tilde{U} and $\pi_\Gamma^{-1}(V)$ with $\tilde{M}(\tilde{U})$, we also have the following.

Corollary 5.6. *For a holomorphic section $s : \mathbb{H}^*/\Gamma \rightarrow M_\Gamma$, there exists to a point $a \in X$ with $D^{-1}(\Gamma)(\{a\}) = \text{Ker}(D)(\{a\})$ such that the induced holomorphic map $s|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{M}(\tilde{U})$ satisfies*

$$s|_{\tilde{U}}(t') = (t', f_{t'}(a))$$

for all $t' \in \tilde{U}$.

Remark. Note that it is not true that all points a of X with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$ define holomorphic sections of $\pi : M \rightarrow B$. See example 6.1.

Example 5.7. Let (X, u) be the flat surface as in example 3.14. Recall that (X, u) is a Riemann surface of type $(2, 0)$ and $\Gamma(X, u) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right\rangle$. It is easy to see that $\text{Ker}(D) = \langle \alpha : z \mapsto z + 1, \beta : z \mapsto -z \rangle$. The candidates of holomorphic sections of the corresponding Veech holomorphic family of Riemann surfaces are the points a with $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$. The candidates are $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(0, 1)$, $(\frac{1}{2}, 1)$, $(\frac{3}{2}, 1)$ in figure 1. They are the Weierstrass points of X and the hyperelliptic involution of X is α^2 .

Example 5.8. Let (X, u) be the flat surface as in example 3.15. Then $\Gamma(X, u) = \langle R_n, T_n \rangle$. It is easy to see that $\text{Ker}(D) = \langle z \mapsto -z \rangle$. Only the points of X which correspond to the center and the vertices of the corresponding polygon satisfy $\text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})$. Hence, the two points are the candidates of the holomorphic sections of the corresponding Veech holomorphic family of Riemann surfaces. Since, one of them is the critical point of (X, u) and the other is not a critical point of (X, u) , if we construct sections locally from the points and extend them globally, the permutation of these two points does not happen. Hence, the

two points define holomorphic sections and they are all the holomorphic sections of the Veech holomorphic family of Riemann surfaces.

6. DIOPHANTINE PROBLEMS

In this section, we consider Diophantine problems on function fields. We associate the Diophantine problems with holomorphic families of Riemann surfaces and their holomorphic sections. At the end of this section, we construct some examples of Diophantine equations on function fields and give all solutions of them.

Let B_0 be a compact Riemann surface. Denote by $M(B_0)$ the field of all meromorphic functions on B_0 . We take an irreducible homogeneous polynomial

$$f(X, Y, Z) = \sum_{i+j+k=N} A_{i,j,k} X^i Y^j Z^k \quad (A_{i,j,k} \in M(B_0)).$$

Problem (Diophantine problem). Find all solutions of $f(X, Y, Z) = 0$ in $\mathbb{P}^2(M(B_0))$.

We see a relation between Diophantine problems and holomorphic families of Riemann surfaces. We set

$$f_t(x, y, z) = \sum_{i+j+k=N} A_{i,j,k}(t) x^i y^j z^k.$$

Since f is irreducible, there exist $t_1, \dots, t_n \in B_0$ such that $f_t(x, y, z) \in \mathbb{C}[x, y, z]$ is irreducible for each $t \in B = B_0 - \{t_1, \dots, t_n\}$. Then, for each $t \in B$,

$$X_t = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : f_t(x, y, z) = 0\}$$

is an algebraic curve, that is, a compact Riemann surface. Set

$$M = \{(t, [x : y : z]) \in B \times \mathbb{P}^2(\mathbb{C}) : f_t(x, y, z) = 0\}$$

and let $\pi : M \rightarrow B$ be the projection. Then the triple (M, π, B) is a holomorphic family of Riemann surfaces. Next, if $[X : Y : Z] \in \mathbb{P}^2(M(B_0))$ is a solution of $f(X, Y, Z) = 0$, then

$$f(X(t), Y(t), Z(t)) = \sum_{i+j+k=N} A_{i,j,k}(t) X(t)^i Y(t)^j Z(t)^k = 0$$

for all $t \in B_0$. This means that $[X(t) : Y(t) : Z(t)] \in X_t$ for all $t \in B$. Hence, the map $B \ni t \mapsto (t, [X(t) : Y(t) : Z(t)]) \in M$ is a holomorphic section of $\pi : M \rightarrow B$. Conversely, if $s : B \rightarrow M$ is a holomorphic section, then $s(t) = (t, [X(t) : Y(t) : Z(t)])$ for some $X, Y, Z \in M(B_0)$ and $[X : Y : Z]$ is a solution of $f(X, Y, Z) = 0$. In conclusion, a Diophantine problem is to find all holomorphic sections of the corresponding holomorphic families of Riemann surfaces.

Example 6.1. Let (X, u) be the flat surface as in example 3.14. Let (M, π, B) be the Veech holomorphic family of Riemann surfaces defined by (X, u) . Set $S = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right]$ and $T = \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right]$. Then X is a Riemann surface of type $(2, 0)$, $\Gamma(X, u) = \langle S, T \rangle$, $\text{Ker}(D) = \langle \alpha : z \mapsto z + 1, \beta : z \mapsto -z \rangle$, α^2 is the hyperelliptic involution of X , and the Weierstrass points of X are $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$, $(0, 1)$, $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 1)$. We denote the set of all Weierstrass points of X by WP . Since $\mathbb{H}/\Gamma(X, u)$ is genus 0 and has 2 punctures and one cone point of order 2, $B = \mathbb{H}^*/\bar{\Gamma}(X, u)$ is a 3-punctured sphere. Recall that $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{\Gamma}(X, u) = R\Gamma(X, u)R^{-1}$.

We describe each fiber $X_t = \pi^{-1}(t)$ ($t \in B$) as an algebraic curve. By construction, $X_t = (X, A_t \circ u)$ for some $A_t \in \mathrm{SL}(2, \mathbb{R})$. Let $T_{A_t} : (X, u) \rightarrow (X, A_t \circ u)$ be the identity map. Note that T_{A_t} is an affine map whose derivative is A_t with respect to the charts of the flat structures. Set $\alpha_t = T_{A_t} \circ \alpha \circ T_{A_t}^{-1}$, $\beta_t = T_{A_t} \circ \beta \circ T_{A_t}^{-1}$ and $WP_t = T_{A_t}(WP)$. Then $K_t = \langle \alpha_t, \beta_t \rangle$ is a group of conformal automorphisms of X_t . The automorphism α_t^2 is also the hyperelliptic involution of X_t and WP_t is the set of all Weierstrass points of X_t . Then $R_t = (X_t - WP_t) / \langle \alpha_t^2 \rangle$ is a Riemann surface of type $(0, 6)$ and has conformal automorphisms α'_t, β'_t induced by α_t, β_t . We may assume that $0, 1, \infty$ are punctures of Y_t , two critical points of X_t are mapped to $0, \infty$ and another Weierstrass point is mapped to 1 . Then α'_t is a Möbius transformation of order 2 which fixes $0, \infty$ and β'_t is a Möbius transformation of order 2 which permutes 0 and ∞ . Therefore, $\alpha'_t(z) = -z$ and $\beta'_t(z) = \lambda_t/z$ for some $\lambda_t \in \mathbb{C} - \{0, \pm 1\}$. This implies that $R_t = \mathbb{C} - \{0, \pm 1, \pm \lambda_t\}$ and

$$X_t = \{(x, y) \in \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - \lambda_t^2)\}.$$

Also, we can write X_t as

$$X_t = \{(x, y) \in \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - 1/\lambda_t^2)\}.$$

Thus, we have the holomorphic map

$$\Psi : \mathbb{H}/\bar{\Gamma}(X, u) \ni t \mapsto \{\lambda_t^2, 1/\lambda_t^2\} \in (\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle.$$

By construction, X_t and $X_{t'}$ are not conformal equivalent for two distinct points $t, t' \in \mathbb{H}/\bar{\Gamma}(X, u)$. This implies that Ψ is injective, and hence conformal map. The map $\phi(z) = \frac{1}{2}(z + 1/z)$ induces a conformal map from $(\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle$ to $\mathbb{C} - \{0, 1\}$. And, $-1 \in \mathbb{C} - \{0, 1\}$ is a cone point of $(\mathbb{C} - \{0, 1\}) / \langle z \mapsto 1/z \rangle$. Hence, $B = \mathbb{H}^*/\bar{\Gamma}(X, u) = \mathbb{C} - \{0, \pm 1\}$. Since the parameter t is in B and $\lambda_t^2, 1/\lambda_t^2$ are in $\mathbb{C} - \{0, \pm 1\} = \phi^{-1}(B)$ which is a double cover of B , if the parameter t moves on B along a closed curve around 1 or -1 , then the descriptions of X_t as above appear reciprocally. (See figure 4.) We change the base space B to $\phi^{-1}(B) = \mathbb{C} - \{0, \pm 1\}$. By calculation, it is easily proved that $\Gamma = \langle S, T^2, TST^{-1} \rangle$ is the subgroup of $\Gamma(X, u)$ which corresponds to the covering map $\phi : \mathbb{C} - \{0, \pm 1\} \rightarrow B$. Let $(M_\Gamma, \pi_\Gamma, \mathbb{C} - \{0, \pm 1\})$ be the holomorphic family of Riemann surfaces induced by (M, π, B) by replacing B to $B_\Gamma = \mathbb{H}^*/R\Gamma R^{-1} = \mathbb{C} - \{0, \pm 1\}$. Then

$$\begin{aligned} M_\Gamma &= \{(t, (x, y)) \in B_\Gamma \times \hat{\mathbb{C}}^2 : y^2 = x(x^2 - 1)(x^2 - t)\} \\ &= \{(t, [X : Y : Z]) \in B_\Gamma \times \mathbb{P}^2(\mathbb{C}) : Y^2 Z^3 = X(X^2 - Z^2)(X^2 - tZ^2)\}. \end{aligned}$$

From corollary 5.6, all holomorphic sections of π_Γ correspond to the points in X which satisfy $D^{-1}(\Gamma)(\{a\}) = \mathrm{Ker}(D)(\{a\})$. By computation, it is easy to see that such points are exactly the Weierstrass points. Hence, the candidates of holomorphic sections are

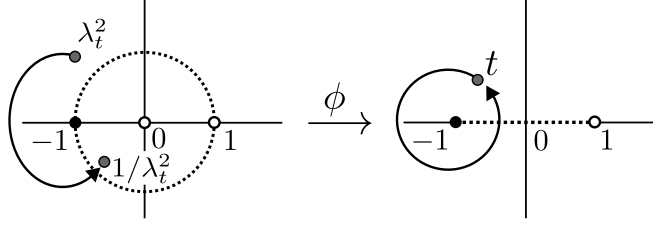
$$\begin{aligned} (t, [X : Y : Z]) &= (t, [0 : 0 : 1]), (t, [1 : 0 : 1]), (t, [-1 : 0 : 1]), \\ &\quad (t, [0 : 1 : 0]), (t, [\sqrt{t} : 0 : 1]), (t, [-\sqrt{t} : 0 : 1]). \end{aligned}$$

They are well-defined except for the last two. Thus, we conclude that all the holomorphic sections of π_Γ are

$$(t, [X : Y : Z]) = (t, [0 : 0 : 1]), (t, [1 : 0 : 1]), (t, [-1 : 0 : 1]), (t, [0 : 1 : 0])$$

and all the solutions of the Diophantine equation $f(X, Y, Z) = X(X^2 - Z^2)(X^2 - tZ^2) - Y^2 Z^3$ in $\mathbb{P}^2(M(\hat{\mathbb{C}}))$ is

$$[X : Y : Z] = [0 : 0 : 1], [1 : 0 : 1], [-1 : 0 : 1], [0 : 1 : 0].$$

FIGURE 4. A closed curve with base point t and a lift of the curve

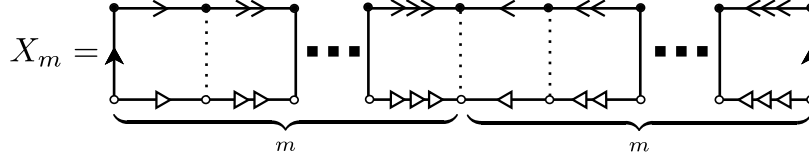
Remark. We can extend $(M_\Gamma, \pi_\Gamma, \mathbb{C} - \{0, \pm 1\})$ to a holomorphic family of Riemann surfaces over $\mathbb{C} - \{0, 1\}$. On the other hand, $t = 0, \infty$ correspond to the Riemann surface with one node obtained by contracting a horizontal closed curve on (X, u) . And $t = 1$ corresponds to a Riemann surface with two nodes obtained by contracting vertical closed curves on (X, u) which are the core curves of two vertical cylinders.

Example 6.2. By the same argument as example 6.1, we obtain the following examples. Let us consider the flat surface (X_m, u_m) as in figure 5 for $m \geq 2$. If m is even, X_m has genus m and two critical points. And if m is odd, X_m has genus $m - 1$ and four critical points. The Veech groups $\Gamma(X_m, u_m)$ are equal to $\Gamma(X, u) = \langle S, T \rangle$ as in example 6.1. Let $(M_m, \pi_m, \mathbb{C} - \{0, \pm 1\})$ be the holomorphic families of Riemann surfaces constructed in the same way as example 6.1. Then $(M_m, \pi_m, \mathbb{C} - \{0, \pm 1\})$ corresponds to a Diophantine equation

$$f_m(X, Y, Z) = X(X^m - Z^m)(X^m - tZ^m) - Y^2 Z^{2m-1} = 0.$$

And all solutions of $f_m(X, Y, Z) = 0$ are

$$[X : Y : Z] = [0 : 0 : 1], [0 : 1 : 0], [\exp(2\pi i \nu / m) : 0 : 1] \quad (\nu = 1, 2, \dots, m).$$

FIGURE 5. The flat surface X_m

7. UPPER BOUND OF THE NUMBER OF THE HOLOMORPHIC SECTIONS

In this section, we give an upper bound of the numbers of holomorphic sections of certain Veech holomorphic families of Riemann surfaces by topological quantities of B and the fiber Riemann surfaces.

Let X be a Riemann surface of type (g, n) and q be an integrable holomorphic quadratic differential on X . Denote by u the flat structure on X induced by q . Let (M, π, B) be the Veech holomorphic family of Riemann surfaces defined by (X, u) . We assume that $\Gamma(X, u)$ is a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$ ($\nu_i \in \{2, 3, \dots, \infty\}$). Then $B = \mathbb{H}^* / \Gamma(X, u)$ is a Riemann surface of type (p, k) . We prove the following theorem.

Theorem 7.1. *If q has a simple Jenkins-Strebel direction, then the number of holomorphic sections of (M, π, B) is at most*

$$32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - 2 + n) - 2g + 2.$$

From corollary 5.5, the number of holomorphic sections is at most the cardinal of the set

$$S(X, u) = \{a \in X : \text{Aff}^+(X, u)(\{a\}) = \text{Ker}(D)(\{a\})\}.$$

We estimate the cardinal of $S(X, u)$. We may assume that $\theta = 0$ is a simple Jenkins-Strebel direction of q . Let $C(X, u)$ be the set of all critical points on X . Let $\varphi : X \rightarrow Y = X/\text{Ker}(D)$ be the branched covering map and $B(X)$ be the set of all branch points of φ . The Riemann surface $Y = X/\text{Ker}(D)$ has the flat structure u' induced by (X, u) . We denote the set of all critical points of (Y, u') by $C(Y, u')$ and the holomorphic quadratic differential on Y which corresponds to (Y, u') by q' . Note that $\varphi(C(X, u)) = C(Y, u')$ is not true. Recall that the Veech group $\Gamma(X, u)$ corresponds to a subgroup of the affine group $\text{Aff}^+(Y, u')$ of (Y, u') . If we consider $\Gamma(X, u)$ as a subgroup of $\text{Aff}^+(Y, u')$, we write $A \in \Gamma(X, u)$ by h'_A . Clearly, $\varphi(S(X, u)) = \{b \in Y : \Gamma(X, u)\{b\} = \{b\}\}$.

Since $\theta = 0$ is a simple Jenkins-Strebel direction, (X, u) is constructed from a rectangle whose vertical sides are glued by a translation and horizontal sides correspond to the union of all horizontal saddle connections of (X, u) . Let $H, W > 0$ be the height and the width of the rectangle, respectively. Let R be the domain in X which corresponds to the interior of the rectangle. Then there exists a chart $z : R \rightarrow (0, W) \times (-\frac{H}{2}, \frac{H}{2})$ in u . For every $h \in \text{Ker}(D)$, there exists an open set $U \subset R$ such that $h(U) \subset R$ and $z \circ h \circ z^{-1} : z(U) \rightarrow z(h(U))$ is the form $z \mapsto \pm z + c$ for some $c \in \mathbb{R}$. The sign of the representation of h is uniquely determined and c is also uniquely determined up to $\pm W$. We identify h with the representation. Then the following lemma is obvious.

Lemma 7.2. *The group $\text{Ker}(D)$ is one of the following form:*

$$\langle z \mapsto z + c \rangle, \langle z \mapsto -z + c \rangle, \langle z \mapsto z + c, z \mapsto -z + d \rangle.$$

Let $\text{sgn} : \text{Ker}(D) \rightarrow \{\pm 1\}$ be the homomorphism which maps $h \in \text{Ker}(D)$ to the derivative of $z \circ h \circ z^{-1}$. Denote by l_y the closed curve in X with $z(R \cap l_y) = (0, W) \times \{y\}$ for $y \in (-\frac{H}{2}, \frac{H}{2})$.

Lemma 7.3. *Every $h \in \text{Ker}(D)$ with $\text{sgn}(h) = -1$ has two fixed points on l_0 and $h^2 = \text{id}$.*

Proof. Since h fixes l_0 and reverse the orientation of l_0 , h has two fixed points in l_0 . This implies that $z \circ h^2 \circ z^{-1}$ is a translation with fixed points in l_0 . Hence, we have $h^2 = \text{id}$. \square

We set

$$\begin{aligned} b_0 &= \inf \left\{ |b| : \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \Gamma(X, u), b \neq 0 \right\}, \\ c_0 &= \inf \left\{ |c| : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, u), c \neq 0 \right\}. \end{aligned}$$

To prove theorem 7.1, we show the following.

Proposition 7.4. *If $\text{sgn}(\text{Ker}(D)) = \{1\}$, then*

$$\#S(X, u) \leq \text{mod}(X) c_0 \left\lceil \frac{b_0 \# \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil - 2g + 2.$$

And if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, then

$$\sharp S(X, u) \leq 4\text{mod}(X)c_0 \lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 2g + 2.$$

Here, $\text{mod}(X) = \frac{W}{H}$ and $\lceil x \rceil$ is the smallest integer which is greater than or equals to x .

By lemmas 7.2 and 7.3, (Y, u') is also constructed from some rectangle R' whose vertical sides are glued by translation and horizontal sides correspond to the union of all horizontal saddle connections of (Y, u') . Denote by W' and H' the width and height of R' , respectively. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, we have $W' = \frac{W}{\sharp \text{Ker}(D)}$ and $H' = H$. If $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, we have $W' = \frac{2W}{\sharp \text{Ker}(D)}$ and $H' = \frac{H}{2}$. Note that, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, the closed curve l_0 in X projects to one of horizontal sides of R' . The horizontal side has two critical points which are poles of q' of order 1. We fix a vertical open interval I in R' which connects two horizontal sides of R' . Let l'_w be the horizontal closed geodesic in Y passing through $w \in I$ and L' be the union of all horizontal saddle connections of Y . We set $h'_{A_0} = A_0 = \left[\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$, $I_0 = \{w \in I : l'_w \text{ is pointwise fixed by } h'_{A_0}\}$, and

$$\text{Cross}(A) = \left(\bigcup_{w \in I_0} (l'_w \cap h'_A(l'_w)) \right) \cup (L' \cap h'_A(L'))$$

for $A = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \Gamma(X, u)$ with $c \neq 0$.

Lemma 7.5. *The sets $\varphi(C(X, u))$, $C(Y, u')$, $\varphi(B(X))$ and $\varphi(S(X, u))$ are contained in $\text{Cross}(A)$.*

Proof. Let $h_A \in \text{Aff}^+(X, u)$ such that $D(h_A) = A$. Since $h_A(C(X, u)) = C(X, u)$, we have $h'_A(\varphi(C(X, u))) = \varphi(C(X, u))$. And $h'_A(C(Y, u')) = C(Y, u')$ because $h'_A \in \text{Aff}^+(Y, u')$. Also, $h_A(B(X)) = B(X)$ implies that $h'_A(\varphi(B(X))) = \varphi(B(X))$. As $\varphi(C(X, u))$, $C(Y, u')$ and $\varphi(B(X))$ are subsets of L' , they are contained in $L' \cap h'_A(L')$. Therefore, $\varphi(C(X, u))$, $C(Y, u')$, $\varphi(B(X)) \subset \text{Cross}(A)$. By the definition of $S(X, u)$, it is clear that $\varphi(S(X, u)) \subset \text{Cross}(A)$. \square

Lemma 7.6. *For all $w \in I$,*

$$\sharp(l'_w \cap h'_A(l'_w)) = \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)}|c| & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)}|c| & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

And,

$$\sharp(L' \cap h'_A(L')) = \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)}|c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)}|c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Here, g' is the genus of Y .

Proof. We identify l'_w with the vector $v = \begin{pmatrix} w' \\ 0 \end{pmatrix}$. Then $h_A(l'_w)$ is identified with $Av = \begin{pmatrix} w'a \\ w'c \end{pmatrix}$. The closed curve $h_A(l'_w)$ pass through l'_w whenever the height

moves H' . Therefore,

$$\#(l'_w \cap h'_A(l'_w)) = W'c/H' = \begin{cases} \frac{\text{mod}(X)}{\#\text{Ker}(D)}|c| & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\#\text{Ker}(D)}|c| & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

For each $p' \in C(Y, u')$, let $\arg(p')$ be the angle around p' . Then $\arg(p') = n\pi$ for some $n \in \mathbb{N}$ and p' appears n times on the horizontal sides of the rectangle R' . Hence,

$$\begin{aligned} \#((L' \cap h'_A(L'))) &= \frac{1}{2} \left(2 \cdot W'c/H' - \sum_{p' \in C(Y, u')} \frac{\arg(p')}{\pi} \right) + \#C(Y, u') \\ &= W'c/H' - \frac{1}{2} \sum_{p' \in C(Y, u')} \left(\frac{\arg(p')}{\pi} - 2 \right) \\ &= W'c/H' - \frac{1}{2} \sum_{p' \in C(Y, u')} \text{ord}(p') \\ &= \begin{cases} \frac{\text{mod}(X)}{\#\text{Ker}(D)}|c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\#\text{Ker}(D)}|c| - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases} \end{aligned}$$

Recall that $\text{ord}(w)$ is the order of w with respect to q' . □

Lemma 7.7.

$$\#I_0 \leq \begin{cases} \lceil \frac{b_0 \# \text{Ker}(D)}{\text{mod}(X)} \rceil & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \lceil \frac{b_0 \# \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 1 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. On the rectangle R' , we can represent h'_{A_0} as the following form:

$$h_{A_0} \begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} W'\xi \\ 0 \end{pmatrix}$$

for some $0 \leq \xi < 1$. If the representation holds by minus, $\#I_0 = 1$. If it holds by plus and $\text{sgn}(\text{Ker}(D)) = \{1\}$, by computation, we have

$$\#I_0 \leq \lceil \frac{b_0 \# \text{Ker}(D)}{\text{mod}(X)} \rceil.$$

If it holds by plus and $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, h'_{A_0} maps each horizontal sides of R' to itself. And one of the sides is the image of the horizontal closed geodesic l_0 and it has exactly two points of $\varphi(B(X))$. Moreover, the length of horizontal saddle connection on Y which connects the two points is $\frac{W'}{2}$. Hence, $\xi = 0$ or $\frac{1}{2}$. By computation, we have

$$\#I_0 \leq \lceil \frac{b_0 \# \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \rceil - 1.$$

□

Proof of proposition 7.4. By lemma 7.6 and lemma 7.7, we have

$$\sharp \text{Cross}(A) \leq \begin{cases} \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil \right) - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ \frac{4\text{mod}(X)}{\sharp \text{Ker}(D)} |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \right\rceil \right) - 2g' + 2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

By lemma 7.5,

$$\sharp S(X, u) \leq \sharp \varphi^{-1}(\text{Cross}(A)) = (\sharp \text{Cross}(A) - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X).$$

Therefore, if $\text{Ker}(D) = \{1\}$,

$$\begin{aligned} \sharp S(X, u) &\leq \text{mod}(X) |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{\text{mod}(X)} + 1 \right\rceil \right) \\ &\quad + (-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X). \end{aligned}$$

And, if $\text{Ker}(D) = \{\pm 1\}$,

$$\begin{aligned} \sharp S(X, u) &\leq 4\text{mod}(X) |c| \left(\left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \right\rceil \right) \\ &\quad + (-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X). \end{aligned}$$

Moreover,

$$\begin{aligned} &(-2g' + 2 - \sharp \varphi(B(X))) \cdot \sharp \text{Ker}(D) + \sharp B(X) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{w \in \varphi(B(X))} (\sharp \text{Ker}(D) - \sharp \varphi^{-1}(w)) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{w \in \varphi(B(X))} \sum_{z \in \varphi^{-1}(w)} (e_z - 1) \\ &= (-2g' + 2) \cdot \sharp \text{Ker}(D) - \sum_{z \in B(X)} (e_z - 1) \\ &= -2g + 2. \end{aligned}$$

Here, e_z is of the ramification index of φ at z and the last equation is the Riemann-Hurwitz formula. Since we can take all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, u)$ with $c \neq 0$, we obtain the claim. \square

Remark. Proposition 7.4 is useful to estimate the number of holomorphic sections if we construct an example of flat surface with a simple Jenkins-Strebel direction. To estimate the number of holomorphic sections, we only need estimating $\sharp \text{Ker}(D)$ and finding some element $A \in \Gamma(X, u)$ with $c \neq 0$. It is easy to see that $\lceil x+1 \rceil \leq 4\lceil \frac{x}{4} + \frac{1}{2} \rceil$ for all $x \in \mathbb{R}$. Hence, the inequality

$$\sharp S(X, u) \leq 4\text{mod}(X) c_0 \left\lceil \frac{b_0 \sharp \text{Ker}(D)}{4\text{mod}(X)} + \frac{1}{2} \right\rceil - 2g + 2$$

always holds. Moreover, we have the following two lemmas.

Assume that $\theta = 0$ is a simple Jenkins-Strebel direction. We can estimate $\sharp \text{Ker}(D)$ by the following way. Let $\theta' (\neq 0)$ be a Jenkins-Strebel direction. Then the rectangle R is decomposed into finitely many parallelograms R_1, R_2, \dots, R_k by the θ' -direction saddle connections. We label the parallelograms with respect to the rule that if R_i and R_j belong the same or congruent cylinders of Jenkins-Strebel direction θ' , then their labels are same. Assume that the labels which we use are

a_1, a_2, \dots, a_l . Let a be the word in the free group $\langle a_1, a_2, \dots, a_l \rangle$ which is obtained by reading the labels from left to right on R . We set

$$n = \max\{n \in \mathbb{N} : a = b^n \text{ for some } b \in \langle a_1, a_2, \dots, a_l \rangle\}.$$

Lemma 7.8. $\sharp \text{sgn}^{-1}(\{1\}) \leq n$.

Proof. Let $h \in \text{sgn}^{-1}(\{1\})$ such that $\text{sgn}^{-1}(\{1\}) = \langle h \rangle$. Since h is a translation, if h sends R_1 to some R_m , then the labels of R_i and R_{i+m} are always same. This implies that $a = b^{\frac{k}{m}}$ for some $b \in \langle a_1, a_2, \dots, a_l \rangle$. Hence, $\sharp \text{sgn}^{-1}(\{1\}) = \frac{k}{m} \leq n$. \square

Lemma 7.9. *If a horizontal closed geodesic is separating curve of X , then*

$$\lceil \frac{b_0 \sharp \text{Ker}(D)}{4 \text{mod}(X)} + \frac{1}{2} \rceil \leq 1.$$

Proof. By the assumption, identifications of two horizontal sides of R are independent. If $\text{sgn}(\text{Ker}(D)) = \{1\}$ then, since there exists the translation $z \mapsto z + \left(\frac{W}{\sharp \text{Ker}(D)} \right)$ in $\text{Ker}(D)$, it is easy to see that $\left[\begin{pmatrix} 1 & \frac{\text{mod}(X)}{\sharp \text{Ker}(D)} \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$. By the definition of b_0 , there exists $m \in \mathbb{N}$ such that $\frac{\text{mod}(X)}{\sharp \text{Ker}(D)} = mb_0$. Hence, $\frac{b_0 \sharp \text{Ker}(D)}{4 \text{mod}(X)} + \frac{1}{2} \leq \frac{1}{4m} + \frac{1}{2}$. By the same argument, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, then $\left[\begin{pmatrix} 1 & \frac{2 \text{mod}(X)}{\sharp \text{Ker}(D)} \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$ and hence, $\frac{b_0 \sharp \text{Ker}(D)}{4 \text{mod}(X)} + \frac{1}{2} \leq \frac{1}{2m} + \frac{1}{2}$ for some $m \in \mathbb{N}$. \square

Now we come back to the proof of theorem 7.1. To prove this, we estimate $\frac{b_0}{\text{mod}(X)}$, $\frac{\text{mod}(X)}{b_0}$, $b_0 c_0$ and $\sharp \text{Ker}(D)$ by g, n, p , and k . Since $\left[\begin{pmatrix} 1 & \text{mod}(X) \\ 0 & 1 \end{pmatrix} \right] \in \Gamma(X, u)$, we have $\frac{b_0}{\text{mod}(X)} \leq 1$ by the definition of b_0 . We set L_0 and L_1 to be the unions of all horizontal saddle connections corresponding to lower and upper horizontal sides of the rectangle R , respectively. Let n_0 be the number of horizontal saddle connections contained in L_0 . And let n_1 be the number of horizontal saddle connections contained in L_1 . We may assume that $n_0 \geq n_1$.

Lemma 7.10. *The inequality*

$$\frac{n_0 + n_1}{2} \leq 2(3g - 3 + n).$$

holds. The equality holds if and only if all punctures of X are poles of q of order 1 and all zeros of q is of order 1.

Proof. By considering the Euler characteristic of X , we have

$$\frac{n_0 + n_1}{2} = 2g - 2 + \sharp C(X, u).$$

Clearly, $\sharp C(X, u) \leq 4g - 4 + 2n$. And, $\sharp C(X, u) = 4g - 4 + 2n$ if and only if all punctures of X are poles of q of order 1 and all zeros of q is of order 1. Hence, we obtain the claim. \square

Lemma 7.11.

$$\frac{\text{mod}(X)}{b_0} \leq \begin{cases} 8(3g - 3 + n)^2 & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ 4(3g - 3 + n)^2 & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. Take $h_{A_0} \in \text{Aff}^+(X, u)$ such that $D(h_{A_0}) = A_0$. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, then there exists $m_0 \leq n_0 + n_1$ such that $h_{A_0}^{m_0}$ fixes L_0 pointwise. And there exists $m_1 \leq n_1$ such that $h_{A_0}^{m_0 m_1}$ fixes L_1 pointwise. Then $m_0 m_1 b_0 = k \cdot \text{mod}(X)$ for some $k \in \mathbb{N}$. Hence, by lemma 7.10,

$$\frac{\text{mod}(X)}{b_0} = \frac{m_0 m_1}{k} \leq \frac{(n_0 + n_1)^2}{2} \leq 8(3g - 3 + n)^2.$$

If $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$, we may assume that $h_{A_0}(L_i) = L_i$ ($i = 0, 1$). There exists $m_0 \leq n_0$ such that $h_{A_0}^{m_0}$ fixes L_0 pointwise. And there exists $m_1 \leq n_1$ such that $h_{A_0}^{m_0 m_1}$ fixes L_1 pointwise. Then $m_0 m_1 b_0 = k \cdot \text{mod}(X)$ for some $k \in \mathbb{N}$. Hence,

$$\frac{\text{mod}(X)}{b_0} = \frac{m_0 m_1}{k} \leq n_0 n_1 \leq \frac{(n_0 + n_1)^2}{4} \leq 4(3g - 3 + n)^2.$$

□

Lemma 7.12.

$$\sharp \text{Ker}(D) \leq \begin{cases} 2(3g - 3 + n) & (\text{sgn}(\text{Ker}(D)) = \{1\}), \\ 4(3g - 3 + n) & (\text{sgn}(\text{Ker}(D)) = \{\pm 1\}). \end{cases}$$

Proof. Let $h \in \text{sgn}^{-1}(\{1\})$. Then $h(L_i) = L_i$ ($i = 0, 1$) and there exists $m \leq n_1$ such that h^m fixes L_1 pointwise. Since h is a translation, $h^m = \text{id}$. Thus,

$$\sharp \text{sgn}^{-1}(\{1\}) \leq m \leq \frac{n_0 + n_1}{2} \leq 2(3g - 3 + n)$$

by lemma 7.10. □

Remark. For the flat surface X_3 as in example 6.2, the equations of lemma 7.10 and lemma 7.12 hold.

Finally, to estimate $b_0 c_0$, we prove a new property of finitely generated Fuchsian groups.

Theorem 7.13. *Let Γ be a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$. Here, $\nu_i \in \{2, \dots, \infty\}$. Assume that k_0 is the number of ν_i 's which are equal to ∞ . If Γ contains $\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right]$ and it is not a power of other elements of Γ , then there exists $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \Gamma$ such that*

$$1 \leq |c| < \text{Area}(\mathbb{H}/\Gamma) - k_0 + 1.$$

Here, $\text{Area}(\mathbb{H}/\Gamma)$ is the hyperbolic area of \mathbb{H}/Γ .

Remark. It is known that

$$\text{Area}(\mathbb{H}/\Gamma) = 2\pi \left(2p - 2 + \sum_{i=1}^k \left(1 - \frac{1}{\nu_i} \right) \right)$$

if Γ is a Fuchsian group of type $(p, k : \nu_1, \dots, \nu_k)$. See [FK92].

Proof. Set $A = \text{Area}(\mathbb{H}/\Gamma) - k_0 + 1$. Assume that $|c| \geq A$ or $c = 0$ for all $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \Gamma$. Set $\Gamma_0 = \left\{ \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \Gamma : c \neq 0 \right\}$. We consider the Ford region D of Γ . Here, the Ford region D is defined by

$$D = \{z \in \mathbb{H} : |\text{Re}(z)| < \frac{1}{2}\} \cap \bigcap_{B \in \Gamma_0} D_B$$

and

$$D_B = \{z \in \mathbb{H} : |z + \frac{d}{c}| > \frac{1}{|c|}\}.$$

By the assumption, D contains the region $D' = \{z \in \mathbb{H} : |\operatorname{Re}(z)| < \frac{1}{2}, \operatorname{Im}(z) > \frac{1}{A}\}$. Let v_1, \dots, v_{k_0} be punctures of \mathbb{H}/Γ . Assume that $\infty \in \overline{\mathbb{H}}$ corresponds to the puncture v_{k_0} . There exists the horodisk U_{k_0} around v_{k_0} which is D' in D . And, by lemma 2.18, there exist horodisks U_i around v_i which are disjoint each other and their hyperbolic areas are 1 for all $i = 1, \dots, k_0 - 1$. Since every domain as figure 7 has area $\pi - 2$, we conclude that $U_i \cap U_{k_0} = \emptyset$ for all $i = 1, \dots, k_0 - 1$. Hence,

$$\begin{aligned} \operatorname{Area}(\mathbb{H}/\Gamma) &> \int_{D'} \frac{dx dy}{y^2} + (k_0 - 1) \cdot 1 \\ &= A + k_0 - 1 \\ &= \operatorname{Area}(\mathbb{H}/\Gamma). \end{aligned}$$

This is a contradiction. □

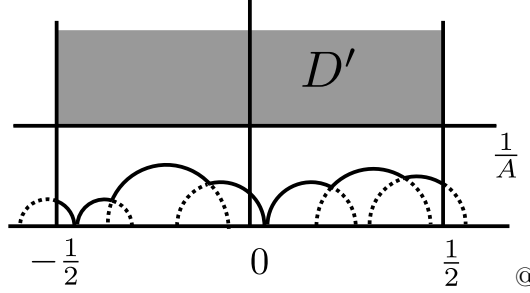


FIGURE 6. The region D' contained in the Ford region D of Γ

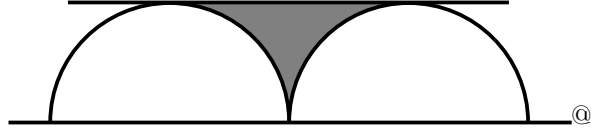


FIGURE 7. A region in \mathbb{H} corresponding to a neighborhood of a puncture of \mathbb{H}/Γ of area $\pi - 2$

Remark. Let Γ be a Fuchsian group of type $(p, k : \infty, \dots, \infty)$ with $k \geq 1$. We suppose that Γ satisfies the assumptions of theorem 7.13. Take the commutator Q of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. Since $\operatorname{tr}(Q) = c^2 + 2 > 2$, theorem 7.13 implies that the Riemann surface \mathbb{H}/Γ has a closed geodesic whose length is less than $2 \cosh^{-1} \left(\frac{(2\pi(2p-2+k)-k+1)^2}{2} + 1 \right)$. If this geodesic is simple, it has a collar neighborhood whose width is greater than $2 \sinh^{-1} \left(\frac{2}{\gamma(p,k) \sqrt{\gamma(p,k)^2 + 4}} \right)$, where $\gamma(p, k) = 2\pi(2p - 2 + k) - k + 1$.

Proof of theorem 7.1. If $\text{sgn}(\text{Ker}(D)) = \{1\}$, by proposition 7.4, lemma 7.11, lemma 7.12, and theorem 7.13,

$$\begin{aligned} \#S(X, u) &\leq 8(3g - 3 + n)^2 \text{Area}(\mathbb{H}/\Gamma(X, u))(6g - 5 + 2n) - 2g + 2 \\ &\leq 32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - \frac{5}{2} + n) - 2g + 2. \end{aligned}$$

And, if $\text{sgn}(\text{Ker}(D)) = \{\pm 1\}$,

$$\begin{aligned} \#S(X, u) &\leq 16(3g - 3 + n)^2 \text{Area}(\mathbb{H}/\Gamma(X, u))(3g - 2 + n) - 2g + 2 \\ &\leq 32\pi(2p - 2 + k)(3g - 3 + n)^2(3g - 2 + n) - 2g + 2. \end{aligned}$$

□

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DEPARTMENT OF MATHEMATICS TOKYO INSTITUTE OF TECHNOLOGY 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO 152-8551, JAPAN

E-mail address: `shinomiya.y.aa@m.titech.ac.jp`